Efficiency Guarantees in Auctions with Budgets

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Abstract

In settings where players have a limited access to liquidity, represented in the form of budget constraints, efficiency maximization has proven to be a challenging goal. In particular, the social welfare cannot be approximated by a better factor then the number of players. Therefore, the literature has mainly resorted to Pareto-efficiency as a way to achieve efficiency in such settings. While successful in some important scenarios, in many settings it is known that either exactly one incentive-compatible auction that always outputs a Pareto-efficient solution, or that no truthful mechanism can always guarantee a Pareto-efficient outcome. Traditionally, impossibility results can be avoided by considering approximations. However, Pareto-efficiency is a binary property (is either satisfied or not), which does not allow for approximations.

In this paper we propose a new notion of efficiency, called *liquid welfare*. This is the maximum amount of revenue an omniscient seller would be able to extract from a certain instance. We explain the intuition behind this objective function and show that it can be 2-approximated by two different auctions. Moreover, we show that no truthful algorithm can guarantee an approximation factor better than 4/3 with respect to the liquid welfare, and provide a truthful auction that attains this bound in a special case.

Importantly, the liquid welfare benchmark also overcomes impossibilities for some settings. While it is impossible to design Pareto-efficient auctions for multi-unit auctions where players have decreasing marginal values, we give a deterministic $O(\log n)$-approximation for the liquid welfare in this setting.

1 Introduction

Budget constraints play a major role in many settings of interest, being internet advertisement one of the most important examples. In fact, the choice of budget to spend is the first question asked to advertisers in the interface of Google Adwords, even before they are asked bids or keywords. Much work has been devoted to understanding the impact of budget constraints in sponsored search auctions. See. e.g., [1, 14, 9]. Budgets are generally important in settings where the magnitude of the financial transactions is very large. Benoit and Krishna [4] point out the importance of budgets in privatization auctions in Eastern Europe and in the sale of spectrum in the US, and more generally in settings where the magnitude of values involved is so large that it might exhaust the players liquid assets. Pai and Vohra [21] illustrate this situation by saying that "not every potential buyer of a David painting who values it at a million dollars has access to a million dollars to make the bid." For a more extensive discussion on the source of financial constraints, we refer to Che and Gale [8].

Due to its practical relevance, it is not surprising that so many theoretical investigations have been devoting to analyzing auctions for budget constrained agents. For analyzing the impact of budget in the revenue of standard auctions one can cite [8] and [4], and for designing mechanims optimize or approximately optimize revenue, one can cite: [16, 18, 21, 6, 7].

When the objective is welfare efficiency rather then revenue, the literature has early stumbled upon impossibility results. The traditional social welfare measure, the sum of player's values for their outcomes, is known not to work well under budget constraints. Indeed, a folklore result shows that even if budgets are known to the auctioneer, no incentive-compatible auction can achieve a better than n approximation for the social welfare, where n is the number of players. This motivates the search for incentive-compatible auctions satisfying weaker notions of efficiency. Dobzinski, Lavi and Nisan [11] suggest studying Pareto-efficient auctions: the outcome of an auction is said to be Pareto-efficient if there it no alternative outcome (allocation and payments) where no agent (bidders or auctioneer) are worse-off and at least one agent is better off. If budgets are public, the authors give an incentive-compatible and Pareto-efficient multi-unit auction based on the Ausubel's clinching framework [2]. Furthermore, they show that this auction is the unique truthful auction that always produces Pareto-efficient solution. The study of Pareto-efficient auctions for budget constrained players has been extended to different settings in a sequence of follow-up papers [5, 13, 9, 14, 15].

Beyond Pareto Efficiency?

It therefore seems that Pareto efficiency has emerged as the de-facto standard for measuring efficiency in settings where bidders are budget constrained. Indeed, most of the aforementioned papers and results provide positive results by offering new auctions. Yet, this is far from being a complete solution from both theoretical and practical point of views. We now elaborate on this issue.

In a sense, the uniqueness result of Dobzinski et al [11] for public budgets – that shows that the clinching auction is the only Pareto efficient, truthful auction – may be viewed negatively. Rare are the cases in practice in which the designer sole goal is to obtain a Pareto efficient allocation. A more realistic view is that theory provides the designer a toolbox of complementary methods and techniques designed to obtain various different goals (efficiency, revenue maximization, fairness, computational efficiency, etc.) and balance between them. The composition of these tools as well as their adaptation to the specifics of the setting and fine tuning is the designer's task. A uniqueness result – although extremely appealing from a pure theoretical perspective – implies that the designer's toolbox contains only one tool, obviously an undesirable scenario.

Furthermore, although from a technical point of view the analysis of the existing algorithms is

very challenging and the proof techniques developed are quite unique to each setting, the auctions themselves are all variants of the same basic clinching idea of Ausubel [2]. Again, it is obviously preferable to have more than just one bunny in the hat that will help us design auctions for these important settings.

The situation is obviously even more severe in more complicated settings, where even this lonely bunny is not available. For example, for private budgets and additive multi-unit auctions, an impossibility was given by [11], for heterogenous items and public budgets by [13, 12] and for as multi-unit auctions with subadditive valuations and public budgets, by Goel, Mirrokni and Paes Leme [14, 17].

Alternatives to Pareto Efficiency

Our main goal is to research alternatives to Pareto efficiency for budget constrained agents. We start from the observation that a Pareto efficient solution is a binary notion: an allocation is either Pareto efficient or not, and there is no sense of one allocation being "more Pareto efficient" than the other. This is in contrast with efficiency in quasi linear environments where the traditional welfare objective induces a total order on the the allocations.

Our main goal in this paper is to provide a new measure of efficiency for budgeted settings. The desiderata for this measure are: (i) it is quantifiable, i.e., attaches a value to each outcome; (ii) is achievable, i.e., can be approximated by incentive-compatible mechanisms and (iii) allows different designs that approximate welfare.

The measure we propose is called the *liquid welfare*. Before defining it, we give a revenue-motivated definition of the traditional social welfare in unbudgeted settings and show how it naturally generalizes to budgeted settings. One can view the traditional welfare of a certain outcome as the maximum revenue an omniscient seller can obtain from that outcome. If each agent i has value $v_i(x_i)$ for a certain outcome x_i, the omniscient seller can extract revenue arbitrarily close to $\sum_i v_i(x_i)$ by offering this outcome to each player i for price $v_i(x_i) - \epsilon$. This definition generalizes naturally to budgeted settings. Given an outcome, x_i, the *willingness-to-pay* of agent i is $v_i(x_i)$, which is the maximum he would give for this outcome in case he had unlimited resources. His *ability-to-pay*, however, is B_i, which is the maximum amount of money available to him. We define his *admissibility-to-pay* as the maximum value he would admit to pay for this outcome, which is the minimum between his willingness-to-pay and his ability-to-pay. The liquid welfare of a certain outcome is defined as the total admissibility-to-pay. Formally $\bar{\mathbf{W}}(x) = \sum_i \min(v_i(x_i), B_i)$.

An alternative point of view is as follows: efficiency should be measured only with respect to the funds available to the bidder at the time of the auction, and not the additional liquidity he might gain after receiving the goods in the auction. When using the liquid welfare objective, the auctioneer is therefore freed from considering the hypothetical use the bidders will make of the items they win in the auction and can focus only on the resources available to them at the time of the auction.

This objective satisfies our first requirement: it associates each outcome with an objective measure. Also, it is achievable. In fact, the clinching auction [11], which is the base for all auction achieving Pareto-efficient outcomes for budgeted settings, provide a 2-approximation for the liquid welfare objective. To show that this allows flexibility in the design, we show a different auction that also provides a 2-approximation and reveals a connection between our liquid welfare objective and the notion of market equilibrium.

It is appropriate to discuss the applicability and limitations of the liquid welfare objective. We would like to begin by illustrating a setting for which it is *not* applicable. If one were to auction hospital beds or access to doctors, it would be morally repugnant to privilege players based on their

ability-to-pay. Therefore, we are not interested in claiming that the liquid welfare objective is the only alternative to Pareto efficiency, but rather argue that in *some* settings it produces reasonable results. Developing other notions of efficiency is an important future direction.

Still, in many settings capping the welfare of the agents by their budgets makes perfect sense. Consider designing a market like internet advertising which aims at a healthy mix of good efficiency and revenue. In practice, players that bring more money to the market provide health to the market and make it more efficient. In real markets, there are practices to encourage wealthier players to enter the market. Therefore, privileging such players in the objective is somewhat natural.

An interesting question is whether one can have a truthful mechanism for additive valuations with public budgets that provides an approximation ratio better than 2 approximation. We show a lower bound of $\frac{4}{3}$. Closing the gap remains an open question, but we do show that for the special case of 2 players with identical public budgets there is a truthful auction that provides a matching upper bound.

We then move one to consider a setting in which truthful auction that always output Pareto-efficient solution do not exist: indivisible multi-unit auctions with additive valuations and private budgets. For this setting we borrow ideas from Bartal, Gonen and Nisan [3] and provide a deterministic $O(\log^2 n)$ approximation to the liquid welfare. In fact, the algorithm is even more powerful since the approximation ratio holds even if the valuations of the bidders are known to be subadditive.

Related Work

We have already surveyed results designing mechanisms for budget-constrained agents. Here, we focus on surveying results directly related to our efficiency measure and to our philosophical approach to efficiency maximization. As far as we know, the liquid welfare was first appeared in Chawla, Malec and Malekian [7] for the first time as an implicit upper bound on the revenue that a mechanism can extract.

Independently and simultaneously, two other approaches were proposed to provide quantitative guarantees for budgeted settings. Devanur, Ha and Hartline [10] show that the welfare of the clinching auction is a 2-approximation to the welfare of the best envy-free equilibrium. Their approach, however, is restricted to settings with *common budgets*, i.e., all agents have the same budget.

Syrgkanis and Tardos [22] leave the realm of truthful mechanisms and study the set of Nash and Bayes-Nash equilibria of simple mechanisms. For a wide class of mechanisms they show that the *traditional* welfare in equilibrium of such mechanism is a constant fraction of the optimal liquid welfare objective (which they call *effective welfare*). Their approach differs from ours in two ways: first they study auctions in equilibrium while we focus on incentive compatible auctions. Second, the guarantee in their mechanism is that the welfare of the allocation obtained is always greater than some fraction of the liquid welfare. The guarantee of our mechanisms is stronger: we construct mechanisms in which the *liquid welfare* is always greater than some fraction of the liquid welfare, which implies in particular that the welfare is greater than some fraction of the liquid welfare (since the welfare of an allocation is at least its liquid welfare).

Summary of Our Results

In this paper we have proposed to study the liquid welfare. We have provided two truthful algorithms that provide a 2 approximation to this objective function for the setting of multi-unit auctions with public budgets. For the harder setting of multi-unit auctions with subadditive valuations and private budgets we have provided a truthful algorithm that provides an $O(\log^2 n)$. For submodular bidders, the same mechanism provides an $O(\log n)$ approximation.

The biggest immediate problem that we leave open is to determine whether there is an algorithm for multi unit auction with private budgets that obtain a constant approximation ratio. The question is even open if the valuations are known to be additive. More generally, are there truthful algorithms that provides a good approximation for *combinatorial* auctions? On top of that, notice that computational considerations might come into play: while all of the constructions that we present in this paper happen to be computationally efficient, there might be a gap between the power of truthful algorithms in general and the power of computationally efficient truthful algorithms.

2 Preliminaries

2.1 Environments of interest and auction basics

We consider n players and a set X of outcomes (also called environment). For each player, let $v_i : X \to \mathbb{R}_+$ be the valuation function for player i. We consider that agents are budgeted quasi-linear, i.e., each agent i has a budget B_i and for an outcome x and for payments $\pi_1, \ldots, \pi_n$, the utility of agent i is: $u_i = v_i(x) - \pi_i$ if $\pi_i \leq B_i$ and $-\infty$ o.w. Below, we list a set of environments we are interested:

1. *Divisible-multi-unit auctions and additive bidders:* $X = \{(x_1, \ldots, x_n); \sum_i x_i = s\}$ for some constant s and $v_i(x) = v_i \cdot x_i$, so we can represent the valuation function of each agent by a single real number $v_i \geq 0$.

2. *Divisible-multi-unit auctions with subadditive bidders:* $X = \{(x_1, \ldots, x_n); \sum_i x_i = s\}$ for some constant s and $v_i(x) = v_i(x_i)$, where $v_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone non-decreasing subadditive function. A valuation function v_i is subadditive if for every x_1, x_2, $v_i(x_1+x_2) \leq v_i(x_1)+v_i(x_2)$.

3. $0/1$ *environments:* $X \subseteq \{0,1\}^n$ and $v_i(x) = v_i$ if $x_i = 1$ and $v_i(x) = 0$ otherwise. Again the valuation is represented by a single $v_i \geq 0$.

An auction for a particular setting elicits the valuations of the players and budgets $B_1, \ldots, B_n$ and outputs an outcome $x \in X$ and payments $\pi_1, \ldots, \pi_n$ for each agents respecting budgets, i.e., such that $\pi_i \leq B_i$ for each agent. We will distinguish between *public budgets* and *private budgets* mechanisms. In the former, the auctioneer has access to the true budget of each agent[1]. In the later case, agents need to be incentivized to report their true budget. In either case, the valuations of each agent are private. We will focus on designing mechanisms that are incentive compatible (a.k.a. truthful), i.e., are such that agents utilities are maximized once they report their true value in the public budget case and their true value and budget in the private budget case. We will also require mechanisms to be individually rational, i.e., agents always derive non-negative utility upon bidding their true value.

In the case of divisible multi-unit auctions and additive bidders, the valuations can be represented by real numbers, So we can see the auctions as a pair of functions $x : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ and $\pi : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ that map (v, B) to a vector of allocations $x(v, B) \in \mathbb{R}_+^n$ and a vector of payments $\pi(v, B) \in \mathbb{R}_+^n$. The set of functions that induce incentive compatible and individually rational auctions are characterized by Myerson's Lemma:

[1] Most of the papers in the literature on efficient auctions for budgeted settings [11, 13, 14, 9, 15] fall in this category, including classical references as, Laffont and Robert [16] and Maskin [19].

Lemma 2.1 (Myerson [20]) *A pair of functions (x, π) define an incentive-compatible and individually rational auction iff (i) for each v_{-i}, $x_i(v_i, v_{-i})$ is monotone non-decreasing in v_i and (ii) the payments are such that: $\pi_i(v_i, v_{-i}) = v_i \cdot x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(u, v_{-i}) du$.*

2.2 Efficiency measures

The traditional efficiency measure in mechanism design is the *social welfare* which associates for each outcome x, the objective: $\mathbf{W}(x) = \sum_i v_i(x)$. It is known that one cannot even approximate the optimal welfare in budgeted settings in an incentive-compatible way, even if the budgets are known and equal. The result is folklore and we only sketch the proof here for completeness:

Lemma 2.2 (folklore) *Consider the divisible-multi-unit auctions and additive bidders. There is no α-approximate, incentive compatible and individually rational mechanism $x(v), \pi(v)$ with $\alpha < n$. For $\alpha = n$ there is the mechanism that allocates the item at random to one player and charges nothing.*

Proof : Let all the agents have budget $B_i = 1$. First notice that $\lim_{v_i \to \infty} x_i(v_i, v_{-i}) \geq \alpha^{-1}$ since $\sum_j v_j \cdot x_j(v) \geq \alpha^{-1} \max_i v_i \geq \alpha^{-1} v_i$, so: $x_i(v) + \sum_{j \neq i} \frac{v_j}{v_i} \cdot x_j(v) \geq \alpha^{-1}$. Taking $v_i \to \infty$ we get that $\lim_{v_i \to \infty} x_i(v_i, v_{-i}) \geq \alpha^{-1}$.

By incentive compatibility, $v_i x_i(v) - \pi_i(v) \geq v_i x_i(v_i', v_{-i}) - \pi_i(v_i', v_{-i})$. Using the fact that $0 \leq \pi_i \leq 1$ we have: $v_i x_i(v) \geq v_i x_i(v_i', v_{-i}) - 1$ so: $x_i(v) \geq x_i(v_i', v_{-i}) - \frac{1}{v_i}$. Taking $v_i' \to \infty$ we get: $x_i(v) \geq \alpha^{-1} - \frac{1}{v_i}$. Summing for all players we get $1 \geq \sum_i x_i(v) \geq n\alpha^{-1} - \sum_i \frac{1}{v_i}$. Taking $v_i \to \infty$ for all i, we get $n\alpha^{-1} \leq 1$. ■

Due to impossibility results of this flavor, efficiency was mainly achieved in the literature through *Pareto efficiency*. We say that an outcome (x, π) with $x \in X$ and $\pi_i \leq B_i$ is Pareto-efficient if there is no alternative outcome where the utility of all the agents involved (including the auctioneer, being his utility the revenue $\sum_i \pi_i$) does not decrease and at least one agent improves. Formally, (x, π) is Pareto optimal iff there is no (x', π'), $x' \in X$, $\pi_i' \leq B_i$ such that:

$$u_i' = v_i \cdot x_i' - \pi_i' \geq u_i = v_i \cdot x_i - \pi_i, \forall i \quad \text{and} \quad \textstyle\sum_i \pi_i' \geq \sum_i \pi_i \quad \text{and} \quad \sum_i v_i x_i' > \sum_i v_i x_i$$

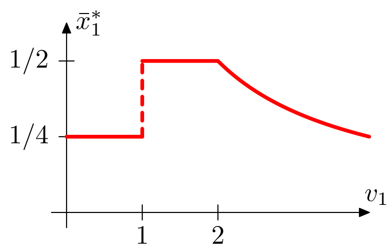
In particular, if the budgets are infinity (or simply very large), then the only Pareto-optimal outcomes are those maximizing social welfare. For the setting of divisible-multi-unit auctions with additive bidders, this is achieved by the Adaptive Clinching Auction of Dobzinski, Lavi and Nisan [11]. Moreover, the authors show that this is the only incentive-compatible, individually-rational auction that achieves Pareto-optimal outcomes. The auction is further analyzed in Bhattacharya et al [5] and Goel et al [15].

In this paper we propose the *liquid welfare* objective function:

Definition 2.3 (Liquid Welfare) *In a budgeted setting, we define the liquid welfare associated with outcome $x \in X$ by $\bar{\mathbf{W}}(x) = \sum_i \min\{v_i(x), B_i\}$.*

We will refer to the optimal liquid welfare as $\bar{\mathbf{W}}^* = \max_{x \in X} \bar{\mathbf{W}}(x)$. It is instructive yet straightforward to see that:

Lemma 2.4 *For divisible-multi-unit auctions and additive bidders, the optimal liquid welfare $\bar{\mathbf{W}}^*$ occurs for $\bar{x}_i^* = \min\left(\frac{B_i}{v_i}, [1 - \sum_{j<i} \bar{x}_j^*]^+\right)$ where players are sorted in non-increasing order of value, i.e., $v_1 \geq v_2 \geq \ldots \geq v_n$.*



$$\bar{x}_1^*(v_1, v_{-1}) = \begin{cases} 1/4, & 0 \leq v_1 \leq 1 \\ 1/2, & 1 \leq v_1 \leq 2 \\ 1/v_1, & 2 \leq v_1 \end{cases}$$

Figure 1: Depiction of the first component of $\bar{x}^* = \text{argmax}_x \bar{\mathbf{W}}(x)$ for a 3 agent instance with $v = (v_1, 1, 2)$ and $B = (1, 1/4, 1)$. The figure highlights the non-monotonicity of the optimal solution $\bar{x}^*(v)$.

An easy observation is the optimal allocation for $\bar{\mathbf{W}}^*$ is not monotone in v_i, and hence cannot be implemented by a truthful auction. For example, consider 3 agents with values $v_1 = v, v_2 = 1, v_3 = 2$ and budgets $B_1 = 1, B_2 = \frac{1}{4}, B_3 = 1$. Now, it is simple to see that $\bar{x}_1^*(v_1)$ is not monotone in v_1 as depicted in Figure 1.

2.3 VCG and the liquid welfare objective

The reader might suspect, however, that a modification of VCG might take care of optimizing the liquid welfare benchmark. This is indeed true for a couple of very simple settings. For example, for selling one indivisible item, a simple Vickrey auction on modified values: $\bar{v}_i = \min\{v_i, B_i\}$ provides an incentive compatible mechanism that exactly optimized the liquid welfare objective. More generally:

Theorem 2.5 (0/1 environments) *Given a $0/1$-environment $X \subseteq \{0,1\}^n$ with valuations $v_i(x) = v_i$ if $x_i = 1$ and zero otherwise. Then running VCG on modified values $\bar{v}_i = \min\{v_i, B_i\}$ is incentive compatible and exactly optimized the liquid welfare objective $\bar{\mathbf{W}}^*$.*

The proof is trivial. This slightly generalizes to other simple environments of interest, for example, matching markets, where there are n agents and n indivible items and each agent i has a value v_{ij} for item j and possible outcomes are perfect matchings. Running VCG on $\bar{v}_{ij} = \min\{v_{ij}, B_i\}$ provides an incentive compatible mechanism that also exactly approximated $\bar{\mathbf{W}}^*$.

This technique, however, does not generalize past those few special cases:

Example. Consider the problem of allocating 1 divisible good among n agents. One possible VCG-reduction is to run VCG assuming that the agents values are $\bar{v}_i(x_i) = \min\{B_i, v_i \cdot x_i\}$. This approach breaks since the resulting mechanism is not monotone (Figure 1) and hence not incentive compatible. Other VCG approach is to assume agents are quasi-linear with values $\bar{v}_i = \min\{v_i, B_i\}$. This approach generates an incentive compatible mechanism, but fails to produce non-trivial approximation guarantees. Consider for example on agent with $B_1 = 2$ and $v_1 = 2$ and other $n-1$ agents with $v_i = 1000$ and $B_i = 1$. The reduction gives an auction between quasi-linear agents with $\bar{v}_1 = 2$ and $\bar{v}_i = 1$ for $i = 2..n$. The auction, therefore, allocates the entire item to player 1, generating liquid welfare $\bar{\mathbf{W}}(x) = 2$, while $\bar{\mathbf{W}}^* = O(n)$. The same example works for allocating a large number of identical indivisible goods. ■

3 A First 2-Approximation: The Clinching Auction

In the previous section we defined our proposal for an efficiency measure in budgeted settings: the liquid welfare objective $\bar{\mathbf{W}}$. The second item in the desiderata for a new efficiency measure is that it is achievable, i.e., it could be optimized or well-approximated by an incentive compatible mechanism. In this section we show a mechanism that provides a 2-approximation for the liquid welfare, while still producing Pareto-efficient outcomes. The mechanism we use is the Adaptive Clinching Auction [11, 5, 15]. In the next section we will provide another truthful auction that provides a 2-approximation, and show that with respect to the liquid welfare the new auction is actually better on an instance-by-instance basis.

We begin by reviewing the auction. The Adaptive Clinching Auction is described by means of an ascending price procedure: we consider a price clock, whose price p starts at zero and gradually ascends. Let's say that the clock ascends in increments of ϵ. Initially, the good is un-allocated and each player has their entire initial budget. For each price, we calculate how much each player would like to acquire at each given price, which is his remaining budget over the price p if the price is below his marginal value. We call such value the *demand*. For each player i we let player i acquire at price p an amount δ_i of the good corresponding to the difference between the un-allocated amount and the sum of demands of the other players. Then δ_i is allocated to player i and his budget is subtracted by $p \cdot \delta_i$. In the limit as ϵ goes to zero, the auction is decribed by means of the following ascending price procedure:

Definition 3.1 (Adaptive Clinching Auction) *Consider one unit of a divisible good to be sold to n agents with valuations $v_1, \ldots, v_n$ per unit and budgets $B_1, \ldots, B_n$. We assume $v_i \neq v_j$ for each $i \neq j$ [2]. The final outcome is calculated by means of an ascending price procedure which can be cast as a differential equation: define functions $x_i(p), \pi(p)$ defined for each $p \in \mathbb{R}_+$. Also, define the auxiliary function $S(p) = 1 - \sum_i x_i(p)$ which represents the* remnant supply *and define the sets $A(p)$ and $C(p)$ which we call the* active set *and the* clinching set *respectively:*

$$A(p) = \{i; p < v_i\} \qquad C(p) = \{i \in A(p); S(p) = \textstyle\sum_{j \in A(p) \setminus i} \frac{B_j - \pi_j(p)}{p}\}$$

Now, $x_i(p)$ and $\pi_i(p)$ are defined by means of the following rules:

- *for $p \notin \{v_1, \ldots, v_n\}$, $x_i(p), \pi_i(p)$ are differentiable and the derivatives for $i \in C(p)$ are given by $\partial_p x_i(p) = S(p)/p$ for $\partial_p \pi_i(p) = S(p)$ and for $i \notin C(p)$ $\partial_p x_i(p) = \partial_p \pi_i(p) = 0$.*
- *given a function f and a price p, we define by $f(p-) = \lim_{x \uparrow p} f(x)$ and $f(p+) = \lim_{x \downarrow p} f(x)$. So, for $p = v_i$, we define: $\delta^i_j = \left[S(v_i-) - \sum_{k \in A(v_i) \setminus j} \frac{B_k - \pi_j(v_i-)}{v_i}\right]^+$ and $x_j(v_i+) = x_j(v_i) = x_j(v_i-) + \delta^i_j$, $\pi_j(v_i+) = \pi_j(v_i) = \pi_j(v_i-) + v_i \cdot \delta^i_j$ for $j \in A(v_i)$ and $x_j(v_i+) = x_j(v_i) = x_j(v_i-)$, $\pi_j(v_i+) = \pi_j(v_i) = \pi_j(v_i-)$ for $j \notin A(v_i)$.*

The final allocation and payments of the clinching auction are given by $\lim_{p \to \infty} x_i(p)$ and $\lim_{p \to \infty} \pi_i(p)$ respectively. Since the values of $x_i(p)$ and $\pi_i(p)$ are constant for $p > \max_i v_i$, the limit is reached for finite price.

Bhattacharya et al [5] show that the procedure described above defines an incentive compatible, individually rational auction and that the outcomes are always Pareto-efficient. Before proving that

[2] If two values are equal, perturb the values to $v_i - \epsilon^i$ and take the limit as $\epsilon \to 0$. We refer to the appendix of Goel, Mirrokni and Paes Leme [15] for a detailed description of the clinching auction when two values are the same without resorting to perturbations of the initial input.

the clinching auction is a good approximation to $\bar{\mathbf{W}}^*$ we define the concept of *clinching interval*, which will be central in our proof. The clinching interval corresponds to to the prices for which the clinching auction allocates goods. Formally:

Definition 3.2 (Clinching Interval) *Given agents with valuations v_i and budgets B_i, let $S(p)$ be the remnant supply function in Definition 3.1. Let $p_0 = \inf\{p; S(p) < 1\}$ and $p_f = \sup\{p; S(p) > 0\}$. We call the interval $[p_0, p_f]$, the clinching interval.*

We will also use the following lemma from Bhattacharya et al [5] (Lemmas 3.3, 3.4 and 3.5 in their paper) :

Lemma 3.3 *If the second highest value is below the budget of the highest value agent, the entire good is allocated to the highest value agent. Otherwise, there is some agent k for which the final price $p_f = v_k$. Moreover the following facts hold:*

- $\forall j, x_j > 0 \Rightarrow v_j \geq v_k$.
- $\forall j, v_j > v_k \Rightarrow \pi_j = B_j$.
- $\forall i, j, v_i < v_k < v_j \Rightarrow \delta^i_j = B_i/v_i$ *and* $\delta^k_j = (B_k - \pi_k)/v_k$.

Now we are ready to state and prove our main Theorem:

Theorem 3.4 *The clinching auction is a 2-approximation to the liquid welfare objective. In other words, given n agents with values per unit v_i and budgets B_i, let x, π be the outcome of the clinching auction for such input. Then, $\bar{\mathbf{W}}(x) \geq \frac{1}{2}\bar{\mathbf{W}}^*$.*

Proof : We assume for simplicity that the values of the players are all distinct. The case where two players have the same value can be handled by perturbing the input and considering the limit on the size of the perturbation. We assume agents are sorted by value, i.e., $v_1 > v_2 > \ldots > v_n$.

In the case where $v_2 \leq B_1$, the clinching auction produces the allocation $x_1 = 1$ according to Lemma 3.3. If $v_1 \leq B_1$, then $\bar{\mathbf{W}}(x) = v_1 = \bar{\mathbf{W}}^*$, otherwise: $\bar{\mathbf{W}}(x) = B_1 \geq \frac{1}{2}(B_1 + v_2) \geq \frac{1}{2}\bar{\mathbf{W}}^*$.

In the remaining case, let x, π be the outcome of the clinching auction and let $[p_0, p_f]$ be the clinching interval. First, we give an upper bound to $\bar{\mathbf{W}}^*$. Let $\bar{x}^*$ be the outcome achieving $\bar{\mathbf{W}}^*$, then:

$$\bar{\mathbf{W}}^* = \sum_i \min(B_i, v_i \cdot \bar{x}^*_i) \leq \sum_{i; v_i > p_0} B_i + \sum_{i; v_i \leq p_0} p_0 \bar{x}^*_i = \sum_{i; v_i > p_0} B_i + p_0 \left[1 - \sum_{i; v_i > p_0} \frac{B_i}{v_i}\right]^+$$

by the format of x^* in Lemma 2.4. Now, we give two lower bounds for $\bar{\mathbf{W}}(x)$ using Lemma 3.3. Let $p_f = v_k$. By our assumption that the players have distinct values, there is only a single player with value p_f.

$$\bar{\mathbf{W}}(x) \geq \sum_i \pi_i = \sum_{i; v_i > p_f} B_i + \pi_k$$

Another way of bounding the revenue of the auction is to consider the ascending price procedure and integrate the derivative over the supply over the price, i.e.:

$$\bar{\mathbf{W}}(x) \geq \sum_i \pi_i = \int_{p_0}^{p_f} p \cdot [-\partial_p S(p)] dp + \sum_{i; v_i \in [p_0, p_f]} v_i \cdot [S(v_i-) - S(v_i)]$$

The first term represents the integral of the price over the derivative of the remnant supply. Since the remnant supply is decreasing, we need to integrate minus the value of the derivative. The sum in the second terms, takes care of the discontinuities in the supply function between v_i- (just before v_i) and v_i. Now, we know by the definition of δ^i_j that $S(v_i-) - S(v_i) = \sum_{j\in C(i)} \delta^i_j$, which gives us:

$$\bar{\mathbf{W}}(x) \geq p_0 \int_{p_0}^{p_f} [-\partial_p S(p)]dp + \sum_{i;v_i\in[p_0,p_f)} v_i \cdot \frac{B_i}{v_i} \cdot |C(v_i)| + v_k \cdot \frac{B_k - \pi_k}{v_k} \cdot |C(v_k)|$$

Now, we note that $\int_{p_0}^{p_f} [-\partial_p S(p)]dp + \sum_{i;v_i\in[p_0,p_f)} \frac{B_i}{v_i} \cdot |C(v_i)| + \frac{B_k-\pi_k}{v_k} \cdot |C(v_k)| = 1$ since it corresponds to the total variation in the supply. So, we have a weighted sum of this total variation, where the weights are all above p_0. From this observation and the fact that $|C(v_i)| \geq 1$ for $v_i \geq p_0$, we can conclude that:

$$\bar{\mathbf{W}}(x) \geq \sum_{i;v_i\in[p_0,p_f)} B_i + (B_k - \pi_k) + p_0 \left[1 - \sum_{i;v_i\in[p_0,p_f)} \frac{B_i}{v_i} - \frac{B_k - \pi_k}{v_k}\right]^+$$

Combining this with the first bound we got on $\bar{\mathbf{W}}(x)$, we get:

$$2\bar{\mathbf{W}}(x) \geq \sum_{i;v_i\geq p_0} B_i + p_0 \left[1 - \sum_{i;v_i\in[p_0,p_f)} \frac{B_i}{v_i} - \frac{B_k - \pi_k}{v_k}\right]^+ \geq \bar{\mathbf{W}}^*$$

by the first bound obtained for $\bar{\mathbf{W}}^*$. ■

A direct consequence from the proof is:

Corollary 3.5 (revenue) *If the clinching auction allocates items to more then one player, then its revenue is at least $\frac{1}{2} \cdot \bar{\mathbf{W}}^*$.*

The following example shows that our analysis is tight:

Example. Consider two agents with $v_1 = 1$, $B_1 = \infty$ and $v_2 = \alpha \gg 1$, $B_2 = 1$ and one unit of a divisible good. For such parameters the allocation $\bar{x}^* = \left(1 - \frac{1}{\alpha}, \frac{1}{\alpha}\right)$ provides the optimal liquid welfare $\bar{\mathbf{W}}^* = 2 - \frac{1}{\alpha}$. The clinching auction generates allocation $x = (0,1)$ for which $\bar{\mathbf{W}}(x) = 1$. As $\alpha \to \infty$, the ratio $\frac{\bar{\mathbf{W}}^*}{\bar{\mathbf{W}}(x)} \to 2$. ■

4 A 2-Approximation via Market Equilibrium

We defined a quantifiable measure of efficiency (Section 2) and showed it can be approximated by an incentive-compatible mechanism (Section 3). The remaining item in the list of desiderata was to show that our efficiency measure allows for different designs. Here we show that we have "an extra bunny in the hat", an auction that also achieves a 2-approximation to the liquid welfare objective and is *not* based on Ausubel's clinching technique. Instead, it is based on the concept of Market Equilibrium.

Borrowing inspiration from general equilibrium theory, consider a market with n buyers each endowed with B_i dollars and willing to pay v_i per unit for a certain divisible good. This is the special case where there is only one product in the market. In this case, a price p is called a *market clearing price* if each buyer can be assigned an optimal basket of goods (in the particular of a single product, an optimal amount of the good) such that there is no surplus or deficiency of any good.

Observe that there is one such price and that allocations can be computed once the price is found. Our Uniform Price Auction simply computes the market clearing price and allocates according to it. This defines the allocation. The payments are computed using the Myerson's formula for this allocation and happen to be different than the clearing price.

Definition 4.1 (Uniform Price Auction) *Consider n agents with values $v_1 \geq \ldots \geq v_n$ (i.e., ordered without loss of generality) and budgets B_i. Consider the auction that allocates one unit of a divisible good in the following way: let k be the maximum integer such that $\sum_{j=1}^k B_j \leq v_k$, then:*

- *Case I: if $\sum_{j=1}^k B_j > v_{k+1}$ allocate $x_i = \frac{B_i}{\sum_{j=1}^k B_j}$ for $i = 1, \ldots, k$ and nothing for the remaining players.*
- *Case II: if $\sum_{j=1}^k B_j \leq v_{k+1}$ allocate $x_i = \frac{B_i}{v_{k+1}}$ for $i = 1, \ldots, k$, $x_{k+1} = 1 - \sum_{j=1}^k x_j$ and nothing for the remaining players.*

Payments are defined through Myerson's integral (Lemma 2.1).

Case I corresponds to the case where the market clearing price of the Fisher Market instance is $p = \sum_{j=1}^k B_j$. Case II coresponds to the case where the Market clearing price is $p = v_{k+1}$. First we show that this auction induces an incentive-compatible auction that does not exceed the budgets of the agents. (Proofs can be found in appendix A.) Then we show that it is a 2-approximation to the liquid welfare benchmark.

Lemma 4.2 (Monotonicity) *The allocation function of the Uniform Price Auction is monotone, i.e., $v_i \mapsto x_i(v_i, v_{-i})$ is non-decreasing.*

Lemma 4.3 (Budget feasibility) *The payments that make this auction incentive-compatible do not exceed the budgets.*

Theorem 4.4 *The Uniform Price Auction is an incentive compatible 2-approximation to the liquid welfare objective.*

Proof : Let k be as in Definition 4.1. First we establish an upper bound on $\bar{\mathbf{W}}^*$. If $\bar{x}^*$ is the allocation achieving $\bar{\mathbf{W}}^*$, then: $\bar{\mathbf{W}}^* = \sum_i \min(v_i \bar{x}_i^*, B_i) \leq \sum_{i=1}^k B_i + \sum_{i=k+1}^n v_i \bar{x}_i^* \leq \sum_{i=1}^k B_i + v_{k+1}$.

Let p be the market clearing price and x the allocation of the Uniform Price Auction. By the definition of k, $p \leq v_i$ for all $i \geq k$, so for those players, $v_i x_i = v_i \frac{B_i}{p} \geq B_i$. Therefore $\bar{\mathbf{W}}(x) \geq \sum_{i=1}^k B_i$. Now, we also show that $\bar{\mathbf{W}}(x) \geq v_{k+1}$. In Case I, this is trivial since: $\bar{\mathbf{W}}(x) \geq \sum_{i=1}^k B_i > v_{k+1}$. In Case II, $v_{k+1} \cdot x_{k+1} = v_{k+1} \cdot \left(1 - \sum_{j=1}^k \frac{B_j}{v_{k+1}}\right) = v_{k+1} - \sum_{j=1}^k B_j < B_{k+1}$, so: $\bar{\mathbf{W}}(x) = \sum_{j=1}^k B_k + v_{k+1} \cdot x_{k+1} = v_{k+1}$.

Summing up two inequalities, we have: $\bar{\mathbf{W}}(x) \geq \frac{1}{2} \left[\sum_{j=1}^k B_j + v_{k+1}\right] \geq \frac{1}{2} \cdot \bar{\mathbf{W}}^*$. ■

The same example used for showing that the analysis for the Clinching Auction was tight can be used for showing that the analysis for the Uniform Price Auction is tight.

Example. Consider two agents with $v_1 = 1$, $B_1 = \infty$ and $v_2 = \alpha \gg 1$, $B_2 = 1$ and one unit of a divisible good. We know that $\bar{\mathbf{W}}^* = 2 - \frac{1}{\alpha}$. The market clearing price for this instance is $p = 1$, which produces an allocation $x = (0, 1)$ with $\bar{\mathbf{W}}(x) = 1$. ■

4.1 Better than the Clinching Auction: An Instance by Instance Comparison

One of the advantages in having a quantifiable measure of efficiency is that we can compare two different outcomes and decide which one is "better". In this section we show that although the worst-case guarantees of the clinching auction and of the uniform-price auction are identical, the liquid welfare of the uniform-price auction is *always* (weakly) dominates that of the clinching auction.

Theorem 4.5 *Consider n players with valuations $v_1 \geq \ldots \geq v_n$ and budgets $B_1, \ldots, B_n$. Let $x^{\mathtt{c}}$ and $x^{\mathtt{u}}$ be the outcomes of the Clinching and Uniform Price Auctions respectively. Then: $\bar{\mathbf{W}}(x^{\mathtt{u}}) \geq \bar{\mathbf{W}}(x^{\mathtt{c}})$.*

Proof : Let $k^{\mathtt{u}}$ be the value of k as in Definition 4.1. We can write the liquid welfare as $\bar{\mathbf{W}}(x^{\mathtt{u}}) = \sum_{j=1}^{k^{\mathtt{u}}} B_j + v_{k^{\mathtt{u}}+1} x^{\mathtt{u}}_{k^{\mathtt{u}}+1}$, where $v_{k^{\mathtt{u}}+1} x^{\mathtt{u}}_{k^{\mathtt{u}}+1} \leq B_{k^{\mathtt{u}}+1}$. Also, for $i \leq k^{\mathtt{u}}$, $B_i = x^{\mathtt{u}}_i \max\{v_{k^{\mathtt{u}}+1}, \sum_{j=1}^{k^{\mathtt{u}}} B_j\} \geq x^{\mathtt{u}}_i v_{k^{\mathtt{u}}+1}$, therefore:

$$1 = \sum_i x^{\mathtt{u}}_i \leq \sum_{i=1}^{k^{\mathtt{u}}} \frac{B_i}{v_{k^{\mathtt{u}}+1}} + x^{\mathtt{u}}_{k^{\mathtt{u}}+1} \qquad (*)$$

Let $\pi^{\mathtt{c}}$ be the payments of the clinching auction for v, B. By Lemma 3.3, there exists $k^{\mathtt{c}}$ such that for every $i \leq k^{\mathtt{c}}$, $\pi^{\mathtt{c}}_i = B_i$ and for $i > k^{\mathtt{c}} + 1$, $x^{\mathtt{c}}_i = 0$. We can write the liquid welfare as: $\bar{\mathbf{W}}(x^{\mathtt{c}}) = \sum_{j=1}^{k^{\mathtt{c}}} B_j + \min\{B_{k^{\mathtt{c}}+1}, v_{k^{\mathtt{c}}+1} \cdot x^{\mathtt{c}}_{k^{\mathtt{c}}+1}\}$. Also, the final price (as in Definition 3.2) is $v_{k^{\mathtt{c}}+1}$, and therefore $B_i = \pi^{\mathtt{c}}_i \leq v_{k^{\mathtt{c}}+1} x^{\mathtt{c}}_i$. Therefore:

$$1 = \sum_i x^{\mathtt{c}}_i \geq \sum_{i=1}^{k^{\mathtt{c}}} \frac{B_i}{v_{k^{\mathtt{c}}+1}} + x^{\mathtt{c}}_{k^{\mathtt{c}}+1} \qquad (**)$$

First we argue that $k^{\mathtt{c}} \leq k^{\mathtt{u}}$. Assume for contradiction that $k^{\mathtt{c}} > k^{\mathtt{u}}$, then:

$$v_{k^{\mathtt{u}}+1} \overset{*}{\leq} \sum_{i=1}^{k^{\mathtt{u}}} B_i + v_{k^{\mathtt{u}}+1} x^{\mathtt{u}}_{k^{\mathtt{u}}+1} \leq \sum_{i=1}^{k^{\mathtt{u}}+1} B_i \leq \sum_{i=1}^{k^{\mathtt{c}}} B_i \overset{**}{\leq} v_{k^{\mathtt{c}}+1}(1 - x^{\mathtt{c}}_{k^{\mathtt{c}}+1}) \leq v_{k^{\mathtt{c}}+1} \leq v_{k^{\mathtt{u}}+1}$$

where the last inequality comes from $k^{\mathtt{c}} > k^{\mathtt{u}}$. This would imply that all inequalities above hold with equalities, in particular: $B_{k^{\mathtt{u}}+1} = v_{k^{\mathtt{u}}+1} x^{\mathtt{u}}_{k^{\mathtt{u}}+1}$, and therefore $x^{\mathtt{u}}_{k^{\mathtt{u}}+1} = \frac{B_{k^{\mathtt{u}}+1}}{v_{k^{\mathtt{u}}+1}}$. Recall that we also have that $x^{\mathtt{u}}_{k^{\mathtt{u}}+1} = 1 - \sum_{i=1}^{k^{\mathtt{u}}} \frac{B_i}{v_{k^{\mathtt{u}}+1}}$, and therefore $v_{k^{\mathtt{u}}+1} = \sum_{j=1}^{k^{\mathtt{u}}+1} B_j$, contradicting the definition of $k^{\mathtt{u}}$.

So, we proved in the previous paragraph that $k^{\mathtt{c}} \leq k^{\mathtt{u}}$. If $k^{\mathtt{c}} < k^{\mathtt{u}}$, then $\bar{\mathbf{W}}(x^{\mathtt{c}}) \leq \sum_{i=1}^{k^{\mathtt{c}}+1} B_i \leq \sum_{i=1}^{k^{\mathtt{u}}} B_i \leq \bar{\mathbf{W}}(x^{\mathtt{u}})$. Now, if $k^{\mathtt{c}} = k^{\mathtt{u}}$, then $(*)$ and $(**)$ together imply that $x^{\mathtt{c}}_{k^{\mathtt{c}}+1} \leq x^{\mathtt{u}}_{k^{\mathtt{u}}+1}$, hence $\bar{\mathbf{W}}(x^{\mathtt{c}}) = \sum_{i=1}^{k^{\mathtt{c}}} B_i + \min\{B_{k^{\mathtt{c}}+1}, v_{k^{\mathtt{c}}+1} \cdot x^{\mathtt{c}}_{k^{\mathtt{c}}+1}\} \leq \sum_{i=1}^{k^{\mathtt{u}}} B_i + \min\{B_{k^{\mathtt{u}}+1}, v_{k^{\mathtt{u}}+1} \cdot x^{\mathtt{u}}_{k^{\mathtt{u}}+1}\} = \bar{\mathbf{W}}(x^{\mathtt{u}})$. ■

5 A Lower Bound and Some Matching Upper Bounds

In the previous sections, we showed two different auctions that are incentive compatible 2-approximations to the optimal liquid welfare for the setting of multi-unit auctions with additive valuations. In this section we investigate the limits of the approximability of the liquid welfare. By the observation depicted in Figure 1, it is clear that an exact incentive compatible mechanism is not possible for this setting. First, we present a $\frac{4}{3}$ lower bound and show matching upper bounds for some special cases.

Theorem 5.1 *For the multi-unit setting with additive values and public budgets, there is no incentive-compatible mechanism that approximates the liquid welfare objective by a factor better then $\frac{4}{3}$.*

Proof : Consider the problem of selling one divisible item to two agents that have equal budgets $B_1 = B_2 = 1$. Let $x(v_1, v_2)$ be the allocation rule for an auction that is a γ-approximation to $\bar{\mathbf{W}}^*$. Also, let $\bar{x}^*(v_1, v_2)$ be solution maximizing the $\bar{\mathbf{W}}$.

Fix some number $\alpha > 1$ and consider the allocations of the auction for valuations $(1, \alpha)$, $(\alpha, 1)$, (α, α). Since the mechanism is truthful, the allocation should be monotone (Lemma 2.1) so: $x_1(1, \alpha) \leq x_1(\alpha, \alpha)$ and $x_2(\alpha, 1) \leq x_2(\alpha, \alpha)$. Since $x_1(\alpha, \alpha) + x_2(\alpha, \alpha) \leq 1$ we get that one of the summands is at most $\frac{1}{2}$. Let us assume that $x_1(\alpha, \alpha) \leq \frac{1}{2}$ and thus $x_1(1, \alpha) \leq \frac{1}{2}$ (the other case is similar). Since $\bar{\mathbf{W}}(\bar{x}^*(1, \alpha)) = \bar{\mathbf{W}}(\bar{x}^*(\alpha, 1)) = 2 - \frac{1}{\alpha}$, the γ approximation implies that:

$$\gamma^{-1}(2 - \tfrac{1}{\alpha}) \leq \min(1, x_1(1, \alpha)) + \min(1, \alpha \cdot x_2(1, \alpha)) \leq x_1(1, \alpha) + 1 \leq 1.5$$

As α approaches ∞ we get that γ approaches $\frac{4}{3}$. ■

5.1 A Matching Upper Bound for 2 Bidders with Equal Budgets

For the special case of 2 players and equal budgets, we give a matching upper bound. Up to re-scaling values and budgets, we can assume that the players have all budgets equal to 1. For this special case, consider the following auction.

Definition 5.2 ($\frac{4}{3}$-approx for $\bar{\mathbf{W}}^*$) *Consider the following auction for* 1 *divisible good and two bidders with (known) budgets $B_1 = B_2 = 1$. It maps values (v_1, v_2) to an allocation $x = (x_1, x_2)$ and is symmetric (i.e., for all v we have that $x_1(v, v) = x_2(v, v)$). So, we only need to specify $x(v_1, v_2)$ for $v_1 \geq v_2$:*

- *if $v_1 = v_2$, $x(v_1, v_2) = (\frac{1}{2}, \frac{1}{2})$*
- *if $v_2 \leq \frac{1}{3}$, $x(v_1, v_2) = (1, 0)$*
- *if $\frac{1}{3} \leq v_2 \leq 1$, $x(v_1, v_2) = (\frac{1}{4} + \frac{1}{4v_2}, \frac{3}{4} - \frac{1}{4v_2})$.*
- *if $1 \leq v_2$, $x(v_1, v_2) = (\frac{1}{2}, \frac{1}{2})$*

Payments are calculated using Myerson's Lemma 2.1.

One can verify that the allocation rule is monotone. Moreover, the payments that make this auction truthful do not exceed the budget, since for any $v_i > 1$, $x_i(v_i, v_{-i}) = x_i(1, v_{-i})$. It remains to prove the approximation guarantee, i.e., that $\bar{\mathbf{W}}(x) \geq \frac{3}{4}\bar{\mathbf{W}}^*$.

Lemma 5.3 *The approximation ratio of the auction in Definition 5.2 is $4/3$, i.e., $\bar{\mathbf{W}}(x) \geq \frac{3}{4}\bar{\mathbf{W}}^*$.*

The proof is by case analysis and can be found in appendix A.

6 An $O(\log^2 n)$-approximation for Subadditive Players with Private Budgets

In this section we consider the setting where players have subadditive valuations and private budgets. This is a notoriously hard setting for Pareto-optimality. In fact, considering either subadditive valuations or privated budgets alone already produces an impossibility result for achieving Pareto-efficient outcomes.

We will have one divisible good and each player has a subadditive valuation $v_i : [0, 1] \to \mathbb{R}_+$ and a budget B_i. This setting differs from the previously considered in the sense that budgets B_i are private information of the players.

The auction we propose is inspired in a technique by Bartal, Gonen and Nisan [3]. To describe it, we use the following notation: $\bar{v}(x_i) = \min\{v_i(x_i), B_i\}$. Now, consider the following selling procedure:

Definition 6.1 (Sell-Without-r) *Let r be a player. Consider the following mechanism to sell half the good, to players $i \neq r$ using the information about $\bar{v}_r(\frac{1}{2})$.*

Divide the segment $[0, \frac{1}{2}]$ into $k = 8\log(n)$ parts, each of size $\frac{1}{2k}$. Associate part $i = 1, \ldots, k$ with price per unit $p_i = \frac{2^i}{8}\bar{v}_r(\frac{1}{2})$. Order arbitrarily all players but player r. Each player different than r, in his turn, takes his most profitable (unallocated) subset of $[0, \frac{1}{2}]$ under the specified prices. Players are not allowed to pay more than their budget.

More precisely, let $p : [0, \frac{1}{2}] \to \mathbb{R}_+$ be such that for $x \in [\frac{1}{2k}(i-1), \frac{1}{2k}i]$, $p(x) = p_i = \frac{2^i}{8}\bar{v}_r(\frac{1}{2})$. Now, for $i = 1, \ldots, r-1, r+1, \ldots, n$, let x_i maximize $v_i(x_i) - \int_{z_i}^{z_i+x_i} p(t)dt$ where $z_i = \sum_{j<i} x_j$, conditioned on the payment being below the budget, i.e., $\int_{z_i}^{z_i+x_i} p(t)dt \leq B_i$. Set the payment as: $\pi_i = \int_{z_i}^{z_i+x_i} p(t)dt$.

The subroutine Sell-Without-r is used in our main construction for this section:

Definition 6.2 (Estimate-and-Price) *Given one divisible good and n players with valuations $v_i(\cdot)$ and budgets B_i, consider the following auction: let $r_1 = \arg\max_i \bar{v}_i(\frac{1}{2})$ and $r_2 = \arg\max_{i \neq r_1} \bar{v}_i(\frac{1}{2})$. We say that r_1 is the* pivot player. *Let (x, π) be the outcome of Sell-Without-r_1 for players $[n] \setminus r_1$ and let (x', π') be the outcome of Sell-Without-r_2 for players $[n] \setminus r_2$.*

For players $i \neq r_1$, allocate x_i and charge π_i. For r_1 if $v_{r_1}(x'_{r_1}) - \pi'_{r_1} \geq v_{r_1}(\frac{1}{2}) - 2 \cdot \bar{v}_{r_2}(\frac{1}{2})$ allocate him x'_{r_1} and charge π'_{r_1} and if not, allocate $\frac{1}{2}$ and charge $2 \cdot \bar{v}_{r_2}(\frac{1}{2})$.

First, notice that the auction defined above is feasible, since r_1 is allocated at most half of the good and the players in $[n] \setminus r_1$ get allocated at most half of the good. Now we show that this auction is incentive compatible:

Lemma 6.3 *The Estimate-and-Price auction is incentive compatible for players with private budgets.*

Proof : We argue that no deviation in which a player changes his value and his budget can be profitable. For a player $i \neq r_1$, if i deviates and does not become the pivot player, his allocation and prices remain the same. Now, this player could try to deviate to become the pivot player. However, in this case he either gets allocated as before, or he is allocated $\frac{1}{2}$ and pays $2\bar{v}_{r_1}(\frac{1}{2}) > \bar{v}_i(\frac{1}{2})$, either exceeding his budget or getting negative utility.

As for the pivot player r_1, he cannot benefit from decreasing $\bar{v}_{r_1}(\frac{1}{2})$ so that he will no longer be the pivot player (all other changes do not change his allocation and payment), since r_2 will become the pivot player in this case and r_1 will be allocated as in the Sell-Without-r_2. This is uselss for r_1: if r_1 was allocated half of the good when playing truthfully, this means that the profit of r_1 could only decrease. If r_1 was allocated as in Sell-Without-r_2 when playing truthfully, then misreporting did not change his allocation and payment. ■

Notice that the price offered to the pivot player is $2 \cdot \bar{v}_{r_2}(\frac{1}{2})$ and not $\bar{v}_{r_2}(\frac{1}{2})$ as the reader who is familiar with [3] would expect. The change is because an auction on one item where every player bids the minimum of his budget and value and the winner pays the second highest bid is

not truthful. To see that, consider two players with identical budgets that their value for the item exceed their budgets. Each of the players has an incentive to report a slight increase in the budget to win the item.

An $O(\log n)$-Approximation for Submodular Bidders

We now show that the Estimate-and-Price auction is a poly-log approximation for subadditive bidders. For clarity of exposition, we first show that for the special case of *submodular bidders*, the Estimate-and-Price auction is a $O(\log n)$ approximation. Then we show how to modify the proof to handle the wider class of subadditive valuations losing only an $O(\log n)$ factor.

Recall that a valuation function v_i is subadditive if $v_i : [0,1] \to \mathbb{R}_+$ such that $v_i(x_i + y_i) \leq v_i(x_i) + v_i(y_i)$. The special case of submodular bidders is characterized by concave v_i functions.

Now, we prove two lemmas towards the proof that the Estimate-and-Price auction is a $O(\log n)$-approximation. First, we claim that not all items are sold in the Sell-Without-r_1 procedure:

Lemma 6.4 *Let x be the outcome of the Sell-Without-r_1 subroutine of the Estimate-and-Price Auction, then $\sum_{i \neq r_1} x_i < \frac{1}{2}$.*

Proof : We note that $\sum_{i\neq r_1} \pi_i \leq \sum_{i\neq r_1} \bar{v}_i(x_i) \leq \sum_{i\neq r_1} \bar{v}_i(\frac{1}{2}) \leq n \cdot \bar{v}_{r_1}(\frac{1}{2})$ by the definition of r_1. Now, if we were to sell $\frac{1}{2}$ to $[n] \setminus r_1$, we would have $\sum_{i\neq r_1} \pi_i = \int_0^{1/2} p(t)dt \geq \int_{k-1/2k}^{1/2} p(t)dt = \frac{1}{2k} \cdot \frac{2^k}{8} \bar{v}_{r_1}(\frac{1}{2}) = \frac{n^8}{16\cdot 8\cdot \log(n)} \cdot \bar{v}_{r_1}(\frac{1}{2}) > n \cdot \bar{v}_{r_1}(\frac{1}{2})$ since for $n \geq 1.8$, $n^7 > 16 \cdot 8 \cdot \log(n)$. ■

Lemma 6.5 *Let $x^\dagger$ be the solution of $\max \bar{\mathbf{W}}(x^\dagger)$ s.t. $x^\dagger_{r_1} = 0$ and $\sum_{i\neq r_1} x^\dagger_i = \frac{1}{2}$, then $\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}(x^\dagger) \geq \frac{1}{2}\bar{\mathbf{W}}^*$.*

Proof : Let x^* be the outcome s.t. $\bar{\mathbf{W}}^* = \bar{\mathbf{W}}(x^*)$, then $\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}(x^\dagger) \geq \bar{\mathbf{W}}(\frac{1}{2}x^*) \geq \frac{1}{2}\bar{\mathbf{W}}(x^*)$, where the first inequality comes from the fact $\frac{1}{2} \geq \frac{1}{2}x^*_{r_1}$ and that $\frac{1}{2}x^*_{-r_1}$ satisfies the requirements of the program defining $x^\dagger$. The second inequality comes from the concavity of $\bar{\mathbf{W}}$. ■

Lemma 6.6 *For submodular bidders, let x be the outcome of the Estimate-and-Price auction and $\bar{p}$ be the price of the cheapest unsold item, i.e., $\bar{p} = \lim_{t\downarrow(\sum_{i\neq r_1} x_i)} p(t)$. Let also x' be any allocation. Then for each $i \neq r_1$, $\bar{v}(x_i) \geq \bar{v}(x'_i) - \bar{p}\cdot x'_i$.*

Proof : If $x_i \geq x'_i$, this is true by monotonicity of $\bar{v}_i$. Now, if not, then player i did not acquire more goods, then it must have been for one of two reasons: either he exhausted his budget or the price exceeds his marginal value. Notice Lemma 6.4 shows that there are items that are available.

In the first case: $\pi_i = B_i$, so $\bar{v}_i(x_i) = B_i \geq \bar{v}_i(x'_i)$. In the second case, $v_i(x_i) \geq v_i(x_i + \epsilon) - p' \cdot \epsilon$ for all $\epsilon \in [0, \epsilon_0]$ for some small ϵ_0. By concavity of $v_i(\cdot)$, $v_i(x'_i) - v_i(x_i) \leq p' \cdot (x'_i - x_i) \leq \bar{p} \cdot x'_i$. So, $\bar{v}_i(x_i) = v_i(x_i) \geq v_i(x'_i) - \bar{p}\cdot x'_i \geq \bar{v}_i(x'_i) - \bar{p}\cdot x'_i$. ■

Now we are ready to prove our main result:

Theorem 6.7 *For submodular bidders, the Estimate-and-Price auction is an $O(\log n)$-approximation to the liquid welfare objective.*

Proof : We analyze two different cases, depending on which fraction of the optimal liquid welfare is produced by player r_1. Let $x^\dagger$ be as in the statement of Lemma 6.5. Then we consider:

Case I: $\bar{v}_{r_1}(\frac{1}{2}) \geq 4\bar{\mathbf{W}}(x^\dagger)$.

We note that the utility of r_1 is at least $v_{r_1}(\frac{1}{2}) - 2 \cdot \bar{v}_{r_2}(\frac{1}{2})$ and that $\bar{v}_{r_2}(\frac{1}{2}) \leq \bar{\mathbf{W}}(x^\dagger)$. Therefore: $u_{r_1} \geq v_{r_1}(\frac{1}{2}) - 2 \cdot \bar{\mathbf{W}}(x^\dagger) \geq \frac{1}{2}\bar{v}_{r_1}(\frac{1}{2}) \geq 2\bar{\mathbf{W}}(x^\dagger)$ which implies that $\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}(x^\dagger) \leq 2u_{r_1} + \frac{1}{2}u_{r_1} = \frac{5}{2}u_{r_1}$. Using Lemma 6.5, we get that: $\bar{\mathbf{W}}(x) \geq \min\{B_{r_1}, u_{r_1}\} \geq \frac{2}{5}[\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}(x^\dagger)] \geq \frac{2}{5}\bar{\mathbf{W}}^*$.

Case II: $\bar{v}_{r_1}(\frac{1}{2}) < 4\bar{\mathbf{W}}(x^\dagger)$.

Consider the price $\bar{p}$ as defined in Lemma 6.6. If $\bar{p} = p_1 = \frac{1}{4}\bar{v}_{r_1}(\frac{1}{2})$, then $\bar{\mathbf{W}}(x) \geq \bar{\mathbf{W}}(x^\dagger) - p_1 \sum_i x_i^\dagger = \bar{\mathbf{W}}(x^\dagger) - \frac{1}{8}\bar{v}_{r_1}(\frac{1}{2}) > \frac{1}{2}\bar{\mathbf{W}}(x^\dagger)$. Therefore: $\bar{\mathbf{W}}^* \leq 2 \cdot [\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}(x^\dagger)] < 10 \cdot \bar{\mathbf{W}}(x^\dagger) < 20 \cdot \bar{\mathbf{W}}(x)$. If on the other hand $\bar{p} = p_k > p_1$, then:

$$\bar{\mathbf{W}}(x) \geq \sum_{i \neq r_1} \pi_i \geq \sum_{i=1}^{k-1} p_i \cdot \frac{1}{2k} \geq (\bar{p} - p_1) \cdot \frac{1}{2k} \geq \frac{1}{2} \cdot \bar{p} \cdot \frac{1}{2k} \geq \frac{\bar{p}}{32 \cdot \log(n)}$$

where the two last inequalities come from the fact that the price doubles every interval. Now, we know that: $16\log(n) \cdot \bar{\mathbf{W}}(x) \geq \frac{1}{2}\bar{p}$. Using that together with Lemma 6.6 for $x_i' = x_i^\dagger$, we get: $(16\log(n) + 1)\bar{\mathbf{W}}(x) \geq \bar{\mathbf{W}}(x^\dagger)$. Since in this case: $\bar{\mathbf{W}}(x^\dagger) \geq \frac{1}{10}\bar{\mathbf{W}}^*$ we have that $\bar{\mathbf{W}}^* \leq O(\log n) \cdot \bar{\mathbf{W}}(x)$. ■

An $O(\log^2 n)$-Approximation for Subadditive Bidders

The only point we used in the previous proof that v_i is concave is in Lemma 6.6. For proving every other argument the subadditivity of v_i suffices. To prove the result for subadditive valuations, we re-define $x^\dagger$ as the argmax of $\bar{\mathbf{W}}(x^\dagger)$ conditioned on $x_i^\dagger = 0$, $\sum_i x_i^\dagger = \frac{1}{2}$ and $x_i^\dagger \leq \frac{1}{2k}$. The extra condition $x_i^\dagger \leq \frac{1}{2k}$ implies that in the proof of Lemma 6.6 no player ever would need to pay more the $2\bar{p}$ per unit. So, we can show that $\bar{v}_i(x_i) \geq \bar{v}_i(x_i^\dagger) - 2\bar{p} \cdot x_i^\dagger$.

The only difference is that under this new definition that gap between $\bar{\mathbf{W}}^*$ and $\bar{v}_{r_1}(\frac{1}{2}) + \bar{\mathbf{W}}^*$ is not constant anymore, but $O(k) = O(\log n)$. This difference makes us lose another $O(\log n)$ factor in the approximation ratio:

Theorem 6.8 *For subadditive bidders, the Estimate-and-Price auction is an $O(\log^2 n)$-approximation to the liquid welfare objective.*

Extension to Indivible Goods

We note that all techniques in this section generalize to indivisible goods. For m identical indivisible goods, one can get the an $O(\log m)$ approximation by dividing the items in $O(\log m)$ groups and setting prices $p_i = 2^i \cdot O(\bar{v}_{r_1}(\frac{1}{2}))$, where $v_{r_1}(\frac{1}{2})$ now stands for the value of player r_1 for half the good.

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A Missing Proofs

Proof of Lemma 4.2 : Fix a valuation profile v with $v_1 \geq \ldots \geq v_n$ and let k be as in Definition 4.1. The agents $i = 1, \ldots, k$ do not change their allocation if they increase their value. For a player $i > k+1$, their allocation can go from zero to non-zero once their value is so high that they become the $k+1$ player. It remains to consider the $k+1$ player: as he increases his allocation and continues to have the $k+1$ highest value, his allocation increases, since it is $x_{k+1} = 1 - \sum_{j=1}^{k} \frac{B_j}{v_{k+1}}$.

Now, two things can happen while v_{k+1} increases:

- the value of v_{k+1} reaches $\sum_{j=1}^{k+1} B_j$ and the allocation gets updated to $\frac{B_{k+1}}{\sum_{j=1}^{k+1} B_j}$. At this point, the allocation of x_{k+1} continues the same as v_{k+1} increases.
- the value of v_{k+1} reaches v_k and displaces k and the k-th highest value. One of two things happen: (i) if $\sum_{j=1}^{k-1} B_j + B_{k+1} > v_k$, then the market clearing price becomes $v_k = v_{k+1}$ and therefore the allocation x_{k+1} gets updated to $1 - \sum_{j=1}^{k-1} \frac{B_j}{v_k} \geq 1 - \sum_{j=1}^{k} \frac{B_j}{v_k}$; or (ii) if $\sum_{j=1}^{k-1} B_j + B_{k+1} \leq v_k$, now the market clearing price becomes v_k and $k+1$ is now allocated as $\frac{B_{k+1}}{v_k} \geq 1 - \sum_{j=1}^{k} \frac{B_j}{v_k}$.

■

Proof of Lemma 4.3 : For players $i > k+1$, this is trivial, since they do not get goods and pay zero. For the rest of the players we look at the two cases:

- Case I : player $k+1$ also does not get goods and pays zero. For the rest of the player, their allocation is constant for any value $v_i' \geq \sum_{j=1}^{k} B_j$, so, their payment is bounded by $(\sum_{j=1}^{k} B_j) \cdot x_i = B_i$.
- Case II: player $k+1$ pays at most $v_{k+1} x_{k+1} = v_{k+1} \cdot \left(1 - \sum_{j=1}^{k} \frac{B_j}{v_{k+1}}\right) < B_{k+1}$, since $v_{k+1} < \sum_{j=1}^{k+1} B_j$ by the definition of k. For the rest of the players $i \leq k$, their allocation is constant for all $v_i' \geq v_{k+1}$, so their payment is bounded by $v_{k+1} \cdot x_i = B_i$.

■

Proof of Lemma 5.3 : The proof is by case analysis. We assume $v_1 \geq v_2$, since $v_2 \geq v_1$ is analogous.

- Case I : $v_1 \geq v_2$ and $v_2 \leq \frac{1}{3}$
  - ∗ I.1 : if $v_1 \leq 1$, then $\bar{\mathbf{W}}(x) = v_1 = \bar{\mathbf{W}}^*$.
  - ∗ I.2 : if $v_1 > 1$, then: $\bar{\mathbf{W}}^* = 1 + \min(1, v_2(1 - \frac{1}{v_1})) \leq 1 + v_2 \leq \frac{4}{3}$ and $\bar{\mathbf{W}}(x) = 1$.

- Case II: $v_1 \geq v_2 \geq 1$

  * II.1: if $v_1 + v_2 \geq 3$, then: $\bar{\mathbf{W}}^* \leq 2$ and $\bar{\mathbf{W}}(x) = \min(1, \frac{1}{2}v_1) + \min(1, \frac{1}{2}v_2) \geq \frac{3}{2}$.
  * II.2: if $v_1+v_2 \leq 3$ then $\bar{\mathbf{W}}^* = 1+\min(1, v_2(1-\frac{1}{v_1})) = 1+v_2(1-\frac{1}{v_1})$ since for $1 \leq v_1, v_2 \leq 2$, it is easy to see that $v_2(1-\frac{1}{v_1}) \leq 1$. Also: $\bar{\mathbf{W}}(x) = \frac{1}{2}(v_1 + v_2)$.

    We want to show that $\frac{1}{2}(v_1 + v_2) \geq \frac{3}{4} \cdot \left[1 + v_2(1 - \frac{1}{v_1})\right]$. Re-writting that, we have that this is equivalent to: $v_2 \geq \frac{2v_1-3}{1-3/v_1}$. Taking derivatives, we see that $\frac{2v_1-3}{1-3/v_1}$ is monotone decreasing in the interval $[1, 2]$, so for this interval: $\frac{2v_1-3}{1-3/v_1} \leq \frac{1}{2} \leq 1 \leq v_2$.

- Case III: $v_1 \geq v_2$, $\frac{1}{3} \leq v_2 \leq 1$

  * III.1: $v_1 \leq 1$, then $\bar{\mathbf{W}}^* = v_1$ and $\bar{\mathbf{W}}(x) = v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2}) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2})$. Therefore, what we want to show is that: $v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2}) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2}) \geq \frac{3}{4}v_1$. We can re-write this as $3v_2 - 1 \geq (2 - \frac{1}{v_2})v_1$. For $\frac{1}{3} \leq v_2 \leq \frac{1}{2}$, this is trivially true by sign analysis, since the left hand side is non-positive and the right hand side is non-negative. For, for $\frac{1}{2} \leq v_2 \leq 1$, this is equivalent to $v_1 \leq \frac{3v_2-1}{2-1/v_2}$, we note that the function $\frac{3v_2-1}{2-1/v_2}$ is monotone non-increasing in the interval $[\frac{1}{2}, 1]$ so: $\frac{3v_2-1}{2-1/v_2} \geq \frac{3}{2} \geq v_1$.
  * III.2: $v_1 \geq 1$ and $1 \leq v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2})$, then: $\bar{\mathbf{W}}^* = 1 + \min(1, v_2(1 - \frac{1}{v_1})) = 1 + v_2(1 - \frac{1}{v_1})$ and $\bar{\mathbf{W}}(x) = \min(1, v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2})) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2}) = 1 + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2})$. Therefore: $\bar{\mathbf{W}}(x) = \frac{3}{4}(1 + v_2) \geq \frac{3}{4}\bar{\mathbf{W}}^*$.
  * III.3: $v_1 \geq 1$ and $1 > v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2})$, then: $\bar{\mathbf{W}}^* == 1 + v_2(1 - \frac{1}{v_1})$ and $\bar{\mathbf{W}}(x) = \min(1, v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2})) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2}) = v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2}) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2})$. We want to show that $v_1 \cdot (\frac{1}{4} + \frac{1}{4v_2}) + v_2 \cdot (\frac{3}{4} - \frac{1}{4v_2}) \geq \frac{3}{4}(1 + v_2(1 - \frac{1}{v_1}))$, which can be re-written as $\frac{v_1}{4} + \frac{v_1}{v_2} \geq 1$, which follows from $v_1 \geq v_2$.

$\blacksquare$